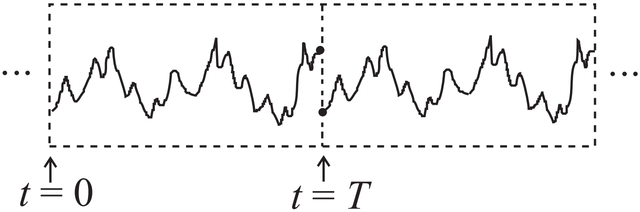
Time series analysis [Book, Ch.3]

4.2 Windows [Book, Sect. 3.2]

When applying Fourier transform to a finite record of duration T, periodicity is assumed for y, which presents a problem.

Unless $y(0)$ and $y(T)$ are of the same value, the periodicity assumption creates a step discontinuity at $y(T)$ (Fig. 1).



The Fourier representation of a step discontinuity requires the use of many spectral components, i.e. spurious energy is leaked to many frequency bands.

Another view is to consider the true time series $Y(t)$ as extending from $-\infty$ to $+\infty$. It is multiplied by a rectangular *window* function

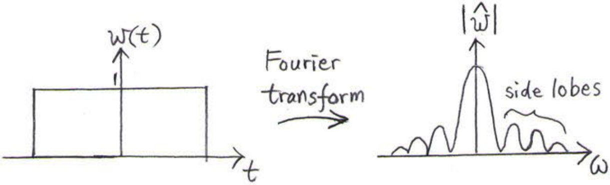
$$w(t) = \begin{cases} 1 & \text{for } -T/2 \leq t \leq T/2 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

to yield the finite data record $y(t)$ of duration T (for convenience, y is now defined for $-T/2 \leq t \leq T/2$).

Thus the data record can be regarded as the product between the true time series and the window function, i.e. $y = wY$.

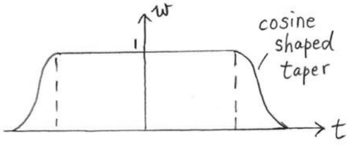
If $\hat{w}$ and $\hat{Y}$ are the Fourier transforms of $w(t)$ and $Y(t)$ over $(-\infty, \infty)$, then the Fourier transform of the product wY is the convolution of $\hat{w}$ and $\hat{Y}$.

For the rectangular window, $\hat{w}$ has many significant side-lobes.

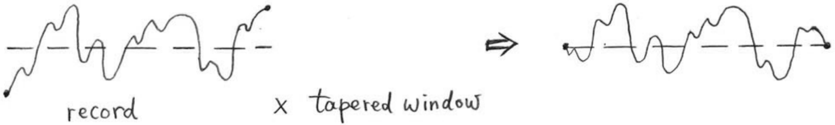


The convolution of $\hat{w}$ and $\hat{Y}$ leads to spurious energy leakage into other frequency bands (Jenkins and Watts, 1968, p.49).

If the ends of the window are tapered (e.g. by a cosine-shaped taper) to avoid the abrupt ends, the size of the side-lobes can be greatly reduced, thereby reducing the spurious energy leakage.



In effect, a tapered window tapers the ends of the data record, so that the two ends continuously approach $y(-T/2) = y(T/2) = 0$, avoiding the original step discontinuity.



Many possible windows (see e.g. Emery and Thomson, 1997).

4.3 Filters [Book, Sect. 3.3]

One often would like to perform digital filtering on the raw data. For instance, one may want a smoother data field, or want to concentrate on the low-frequency or high-frequency signals in the time series.

Express a time series $x(t)$ in terms of its complex Fourier components $X(\omega)$:

$$x(t) = \sum_{\omega} X(\omega)\, e^{i\omega t}\,, \tag{2}$$

where it is understood that ω and t denote the discrete variables ω_m and t_n. A filtered time series is given by

$$\tilde{x}(t) = \sum_{\omega} f(\omega) X(\omega)\, e^{i\omega t}\,, \tag{3}$$

where $f(\omega)$ is the filter "response" function.

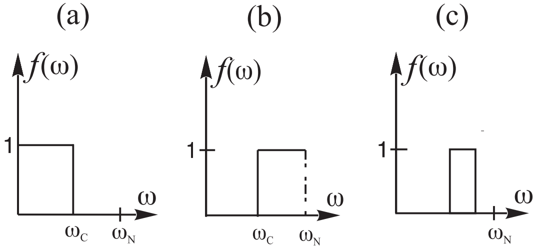


Figure : Ideal filters: (a) low-pass, (b) high-pass, and (c) band-pass, where $f(\omega)$ is the filter "response function", and ω_N is the Nyquist frequency.

In ideal filters, the step discontinuity at the cut-off frequency ω_c produces "ringing" (i.e. oscillations) in the filtered time series (especially at the two ends) (Emery and Thomson, 1997, Fig. 5.10.19).

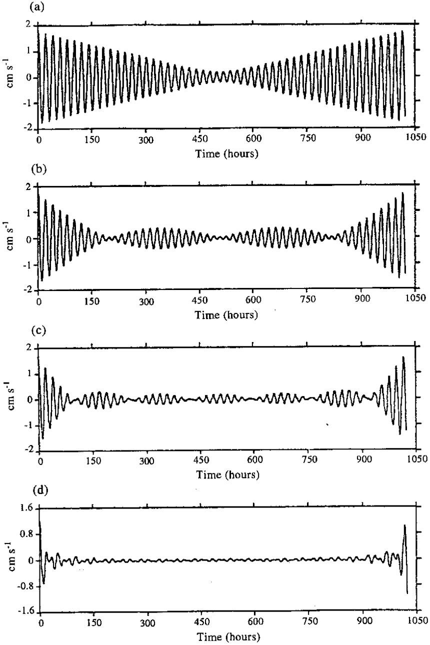


Figure 5.10.19. Ringing effects following application of different discrete Fourier transform filters to an artificial time series with frequency $f = 0.05$ cph and then inverting the transform. (a) Single Fourier coefficient at 0.05 cph set to zero; (b) three Fourier coefficients set to zero; (c) five Fourier coefficients set to zero; (d) 21 coefficients set to zero. (From Forbes, 1988.)

This problem of a step discontinuity in the frequency domain leading to ringing in the time domain mirrors the one mentioned in the previous section, where a time series truncated by a rectangular window lead to energy leakage in the frequency domain.

In practice, $f(\omega)$ needs to be tapered at ω_c to suppress ringing in the filtered time series.

To perform filtering in the *frequency domain*, the steps are:
(i) Fourier transform $x(t)$ to $X(\omega)$,
(ii) multiply $X(\omega)$ by $f(\omega)$,
(iii) inverse transform $f(\omega)X(\omega)$ to get $\tilde{x}(t)$, the filtered time series.

Alternatively, filtering can be performed in the *time domain* as well.

Before the invention of fast Fourier transform algorithms, filtering in the frequency domain was prohibitively expensive.
Nowadays, filtering can be performed in either the frequency or the time domain.

A commonly used time domain filter is the 3-point moving average (or running mean) filter

$$\tilde{x}_n = \frac{1}{3}x_{n-1} + \frac{1}{3}x_n + \frac{1}{3}x_{n+1}\,, \qquad (4)$$

i.e. average over the immediate neighbours.

More generally, a filtered time series is given by

$$\tilde{x}_n = \sum_{l=-L}^{L} w_l\, x_{n+l}\,, \qquad (5)$$

where w_l are the weights of the filter.

Suppose the filtered time series has the Fourier decomposition

$$\tilde{x}_n = \sum_{\omega} \tilde{X}(\omega)\, e^{i\omega t_n}\,. \tag{6}$$

Comparing with (3), one sees that

$$\tilde{X}(\omega) = f(\omega) X(\omega)\,. \tag{7}$$

Thus

$$f(\omega) = \tilde{X}(\omega)/X(\omega) \;=\; \frac{\sum_l w_l\, e^{i\omega l \Delta t} X(\omega)}{X(\omega)}\,, \tag{8}$$

where we have used the fact the Fourier transform is linear, so $\tilde{X}(\omega)$ is simply a linear combination of w_l times the Fourier transform of x_{n+l}.

With $t_l = l\Delta t$,

$$f(\omega) = \sum_{l=-L}^{L} w_l \, e^{i\omega t_l} \,, \tag{9}$$

which allows us to calculate the filter response function $f(\omega)$ from the given weights of a time domain filter.

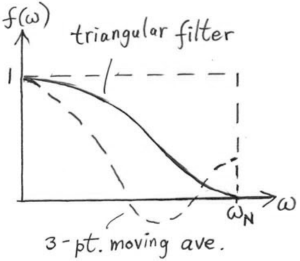
E.g., moving average filters have the general form

$$\tilde{x}_n = \sum_{l=-L}^{L} \left(\frac{1}{2L+1} \right) x_{n+l} \,. \tag{10}$$

Another commonly used filter is the 3-point triangular filter,

$$\tilde{x}_n = \frac{1}{4} x_{n-1} + \frac{1}{2} x_n + \frac{1}{4} x_{n+1} \,, \tag{11}$$

which is better than the 3-point moving average filter in removing grid-scale noise (Emery and Thomson, 1997).



Q2: Suppose the data are dominated by grid-scale noise, i.e. the data in 1-dimension have adjacent grid points simply flipping signs like $\ldots, -1, +1, -1, +1, -1, +1, \ldots$, what happens (a) when you apply the 3-point moving-average filter to the data once? twice? (b) when you apply the triangular filter to the data once? twice?

One often encounters time series containing strong periodic signals, e.g. the seasonal cycle or tidal cycles. While these periodic signals are important, it is often the non-periodic signals which have the most impact on humans, as they produce the unexpected events. One often would remove the strong periodic signals from the time series first.

Suppose one has monthly data for a variable x, and one would like to extract the seasonal cycle. Average all x values in January to get $\overline{x}_{\mathrm{jan}}$, and similarly for the other months. The *climatological seasonal cycle* is then given by

$$\overline{x}_{\mathrm{seasonal}} = [\overline{x}_{\mathrm{jan}}, \cdots, \overline{x}_{\mathrm{dec}}]\,. \tag{12}$$

The filtered time series is obtained by subtracting this climatological seasonal cycle from the raw data— i.e. all January values of x will have $\overline{x}_{\mathrm{jan}}$ subtracted, and similarly for the other months.

For tidal cycles, *harmonic analysis* is commonly used to extract the tidal cycles from a record of duration T. The tidal frequencies ω_n are known from astronomy, and one assumes the tidal signals are sinusoidal functions of amplitude A_n and phase θ_n. The best fit of the tidal cycle to the data is obtained by a least squares fit, i.e. minimize

$$\int_0^T [x(t) - \sum_n A_n \cos(\omega_n t + \theta_n)]^2 \, dt \,, \tag{13}$$

by finding the optimal values of A_n and θ_n.

If T is short, then tidal components with close related frequencies cannot be separately resolved.

A time series with the tides filtered is given by

$$\tilde{x}(t) = x(t) - \sum_n A_n \cos(\omega_n t + \theta_n)\,. \qquad (14)$$

References:

Emery, W. J. and Thomson, R. E. (1997). *Data Analysis Methods in Physical Oceanography*. Pergamon, Oxford.

Jenkins, G. M. and Watts, D. G. (1968). *Spectral Analysis and Its Applications*. Holden-Day, San Francisco.