Chapter 3 lecture questions

Q1: Prove that $\mathbf{M}_f$ and $\mathbf{M}_g$ are positive semi-definite symmetric matrices.

Solution:

Ignoring subscripts for brevity, we write $\mathbf{M}_f = \mathbf{C}\mathbf{C}^{\mathrm{T}}$. Using the identity $(\mathbf{A}\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$,

$$\mathbf{M}_f^{\mathrm{T}} = (\mathbf{C}\mathbf{C}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{C}^{\mathrm{T}})^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} = \mathbf{C}\mathbf{C}^{\mathrm{T}} = \mathbf{M}_f, \tag{1}$$

so $\mathbf{M}_f$ is a symmetric matrix.

To prove it is a positive semi-definite matrix, for any nonzero vector $\mathbf{v}$,

$$\mathbf{v}^{\mathrm{T}}\mathbf{M}_f\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{C}^{\mathrm{T}}\mathbf{v} = (\mathbf{C}^{\mathrm{T}}\mathbf{v})^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{v} = \mathbf{a}^{\mathrm{T}}\mathbf{a} = \|\mathbf{a}\|^2 \geq 0, \tag{2}$$

where $\mathbf{a} = \mathbf{C}^{\mathrm{T}}\mathbf{v}$. Hence $\mathbf{M}_f$ is a positive semi-definite, symmetric matrix. Proof for $\mathbf{M}_g$ is similar.