Chapter 2 lecture questions

Q1: "Prove that $\mathbf{C}$ is a real, symmetric, positive semi-definite matrix" requires us to prove that for any vector $\mathbf{v} \neq \mathbf{0}$, it follows that $\mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{v} \geq 0$.

Proof:

$$\mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathrm{E}[(\mathbf{y} - \overline{\mathbf{y}})(\mathbf{y} - \overline{\mathbf{y}})^{\mathrm{T}}]\mathbf{v}$$

Since $\mathbf{v}$ is a given vector, it can be moved to inside the E[...] operator, i.e.

$$\mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{v} = \mathrm{E}[\mathbf{v}^{\mathrm{T}}(\mathbf{y} - \overline{\mathbf{y}})(\mathbf{y} - \overline{\mathbf{y}})^{\mathrm{T}}\mathbf{v}]$$

This is of the same form as

$$\mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{v} = \mathrm{E}[\mathbf{z}^{\mathrm{T}}\mathbf{z}],$$

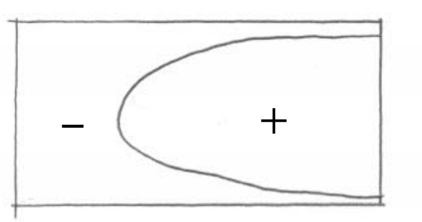
where $\mathbf{z} = (\mathbf{y} - \overline{\mathbf{y}})^{\mathrm{T}}\mathbf{v}$ and $\mathbf{z}^{\mathrm{T}} = \mathbf{v}^{\mathrm{T}}(\mathbf{y} - \overline{\mathbf{y}})$. But $\mathbf{z}^{\mathrm{T}}\mathbf{z} = |\mathbf{z}|^2 \geq 0$, as the length of the vector $\mathbf{z}$ cannot be negative, hence $\mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{v} = \mathrm{E}[\mathbf{z}^{\mathrm{T}}\mathbf{z}] \geq 0$.

Q2: From the principal components during 1984–1991, determine which year the strongest plume of cool water was found to extend offshore from (a) Brooks Peninsula and (b) Cape Scott (at the northern tip of Vancouver Island)?

Answer: (a) 1985 , (b) 1989.
Solution:
From the EOF spatial patterns, we note that cool water (negative SST anomalies) extending offshore from Brooks Peninsula is mainly manifested by mode 3, while cool water extending offshore from Cape Scott, by mode 4. Then we look at the PC time series for mode 3 and mode 4 and see which years when the PCs have the strongest positive values, and the years turned out to be 1985 and 1989, respectively.

Q3: Suppose the first PCA mode has the following EOF spatial pattern and PC time series. You want the EOF to have positive anomalies on the left side instead of on the right

EOF



PC

side. What would you do to achieve this and what would the new PC look like?

Solution:

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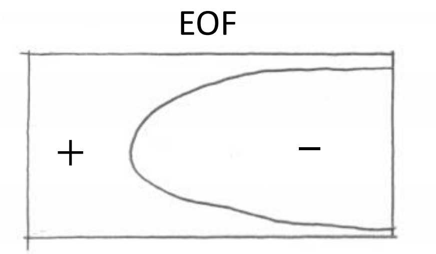
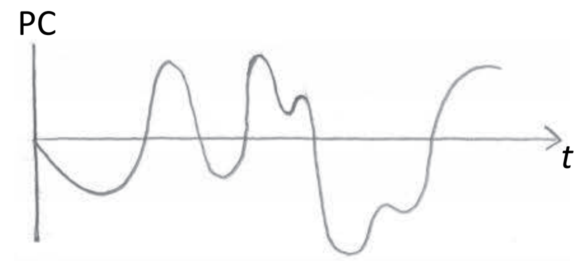
You can simply reverse the sign of the EOF spatial pattern and the sign of the PC, so the new EOF and PC look like:





Q4: Show that with $\mathbf{y} - \overline{\mathbf{y}} = \sum_j a_j \mathbf{e}_j$ and $\mathbf{e}_i^{\mathrm{T}} \mathbf{e}_j = \delta_{ij}$, the variance of the original data $\mathbf{y}$ is contained in $\{a_j(t)\}$, with

$$\mathrm{var}(\mathbf{y}) = \mathrm{E}\left[\|\mathbf{y} - \overline{\mathbf{y}}\|^2\right] = \mathrm{E}\left[\sum_{j=1}^{m} a_j^2\right] . \tag{1}$$

Solution:

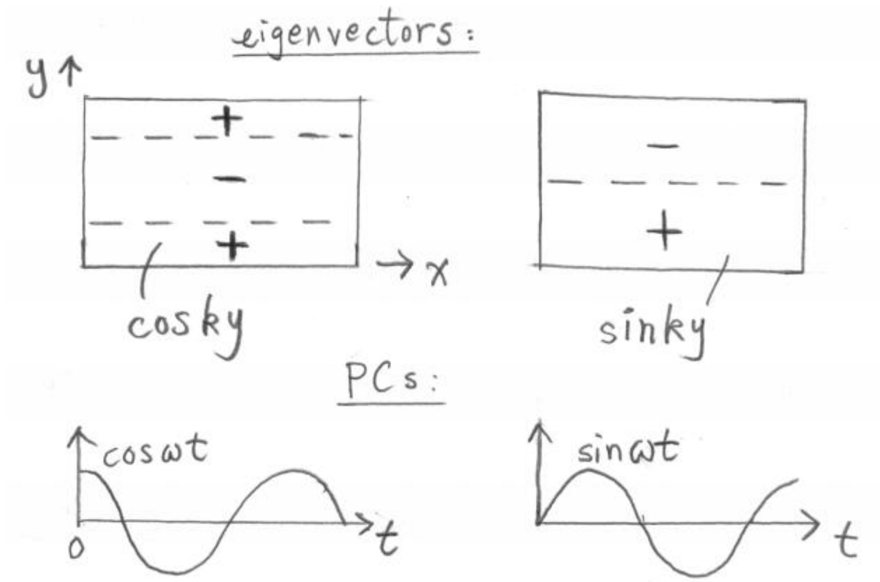
$$\mathrm{E}\left[\|\mathbf{y} - \overline{\mathbf{y}}\|^2\right] = \mathrm{E}\left[(\mathbf{y} - \overline{\mathbf{y}})^{\mathrm{T}}(\mathbf{y} - \overline{\mathbf{y}})\right] = \mathrm{E}\left[\sum_i \sum_j a_i \mathbf{e}_i^{\mathrm{T}} \mathbf{e}_j a_j\right] = \mathrm{E}\left[\sum_i \sum_j a_i \delta_{ij} a_j\right] = \mathrm{E}\left[\sum_{j=1}^{m} a_j^2\right]$$

Q5: If we apply PCA to the propagating plane wave, what do the two EOF spatial patterns look like and what do the corresponding PCs look like?

Solution:
From the lecture, we showed that the propagating plane wave can be written in the form:

$$h = A\cos(ky)\cos(\omega t) + A\sin(ky)\sin(\omega t) . \tag{2}$$

In the x-y plane, $\cos(ky)$ and $\sin(ky)$ are orthogonal, while $\cos(\omega t)$ and $\sin(\omega t)$ are uncorrelated, so (2) satisfies the properties of PCA modes (i.e. eigenvectors are orthogonal and PCs are uncorrelated). Hence the spatial eigenvector (i.e. EOF) patterns are $\cos(ky)$ and $\sin(ky)$, and the corresponding PC time series are $\cos(\omega t)$ and $\sin(\omega t)$ respectively. (See Fig.)



Q6: Which of the following matrices are orthonormal?
(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and (b) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Answer: Both matrices are orthonormal.
Solution:
To show a matrix $\mathbf{M}$ is orthonormal, we need to show that the product $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ gives the identity matrix.
(a) $\mathbf{M}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{M}^{\mathrm{T}}\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(b) $\mathbf{M}^{\mathrm{T}}\mathbf{M} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$
$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q7: Complex PCA is used to analyze the horizontal wind vectors from two stations, with the first station located to the west of the second station. The first eigenvector gives $\mathbf{e}_1^{\mathrm{T}} = [1+\mathrm{i},\ -\mathrm{i}]$. Sketch the horizontal wind field at time (a) $t = 1$ when the first principal component takes on the value -1, and at (b) $t = 2$, when the first PC $= -\mathrm{i}$.

Solution:
$\mathbf{e} = \begin{bmatrix} 1+\mathrm{i} \\ -\mathrm{i} \end{bmatrix}$, and the wind field $\mathbf{y}$ due to the first complex PCA mode is $\mathbf{y}(t) = \mathbf{e}_1\mathbf{a}_1^*(t)$.

(a) At $t = 1$, $\mathbf{a}_1(t) = -1$ and $\mathbf{a}_1^*(t) = -1$, where * denotes complex conjugation. So

$\mathbf{y}(1) = \begin{bmatrix} 1+\mathrm{i} \\ -\mathrm{i} \end{bmatrix} (-1) = \begin{bmatrix} -1-\mathrm{i} \\ \mathrm{i} \end{bmatrix}$, and the wind field is sketched below.

(b) At $t = 2$, $\mathbf{a}_1(t) = -\mathrm{i}$ and $\mathbf{a}_1^*(t) = \mathrm{i}$. So
$\mathbf{y}(1) = \begin{bmatrix} 1+\mathrm{i} \\ -\mathrm{i} \end{bmatrix} \mathrm{i} = \begin{bmatrix} -1+\mathrm{i} \\ 1 \end{bmatrix}$, and the wind field is sketched below.